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t −PEBBLING ON ZIG-ZAG CHAIN GRAPH OF EVEN CYCLES

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Abstract

A pebbling move on a connected graph G consists of removing two pebbles from some vertex and adding one on an adjacent vertex. The $t-$ pebbling number of a connected graph G, denoted by $f_t(G)$, such that any distribution of $f_t(G)$ pebbles on G, we can move t pebbles to any specified vertex by a sequence of pebbling moves. A graph G satisfies the 2t- pebbling property if $2t$ pebbles can be moved to a specified vertex when the total number of pebbles is $2f_t(G) - q + 1$ where q is the number of vertices with at least one pebble. In this paper, we determine the $t-$ pebbling number of the zig-zag chain graph of even cycles and show that it satisfies the $2t-$ pebbling property.

Keywords: Graph pebbling, Zig-zag chain graph, cycle, $2t$ −pebbling property.

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1. Introduction

All graphs in this paper are assumed to be simple, finite, and undirected. For graph theoretic terminologies we follow the notion of [1].

A recent development in graph theory, the pebbling game was first suggested by Lagarias and Saks as a tool for solving number-theoretical conjecture of Erdos. Chung used this tool to establish the results concerning pebbling number. In doing so she introduced the pebbling to the literature [2].

Consider a simple graph G with vertex set $V(G)$ and edge set $E(G)$. For a graph G, let D be a distribution of pebbles on the vertices of G. For any vertex v of G, p_v denotes the number of pebbles on v in D. For $K \subseteq V(G)$, we denote $p(K) = \sum_{v \in V(K)} p(v)$. A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble on an adjacent vertex. In this paper, the letter v will frequently be used to denote the specified vertex of the graph under consideration.

Definition 1.1 [2][3]. The *pebbling number of a vertex* v in G is the smallest number $f(G,v)$ such that every placement of $f(G,v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Also, we define the $t-$ *pebbling number of* v in G is the smallest number $f_t(G,v)$ such that from every placement of $f_t(G,v)$ pebbles, it is possible to move t pebbles to v.

The *pebbling number* of G and the t-*pebbling number* of G are the smallest numbers, $f(G)$ and $f_t(G)$, such that from any placement of $f(G)$ pebbles or $f_t(G)$ pebbles, respectively, it is possible to move one or t pebbles, respectively, to any specified target vertex by a sequence of pebbling moves. Thus, $f(G)$ and $f_t(G)$ are the maximum values of $f(G,v)$ and $f_t(G,v)$ over all vertices v.

Implicit in this definition is the fact that if after moving to a specified vertex v one desires to move to another specified vertex, the pebbles reset to their original initial distribution. Now, we state some facts from [3] about $f(G)$.

1. For any vertex v of a graph $G, f(G,v) \geq n$ where $n = |V(G)|$.
2. The pebbling number of a graph G satisfies $f(G) \geq max\{2^{diam(G)}, |V(G)|\}$, where $diam(G)$ is the diameter of the graph G.

Chung [2] also defined the *two-pebbling property* of a graph, and further Lourdusamy [4] extended her definition to the $2t$ *−pebbling property* as follows.

Definition 1.2 Let D be a distribution of pebbles on G, let q be the number of vertices with at least one pebble. We say that G satisfies the $2-$ *pebbling property* if for any distribution with more than $2f(G)-q$ pebbles, it is possible to move two pebbles to any specified vertex. Further, we say that a graph G has the $2t$*-pebbling property*, if for any distribution with more than $2f_t(G) - q$ pebbles, it is possible to move $2t$ pebbles to any specified vertex.

With regard to t-pebbling number of graphs, we find the following theorems:

Theorem 1.3. [4] *Let P_n be the path with n vertices. Then*

1. $f_t(P_n) = t2^{n-1}$ *and*
2. P_n *satisfies the* $2t-$ *pebbling property.*

Theorem 1.4. *[5][6] Let C_n denote a simple cycle with n vertices, where $n \geq 3$. Then*

(1) $$f_t(C_{2k}) = \begin{cases} t2^k\ , & n(=2k) \text{ is even} \\ \frac{2^{k+2}-(-1)^{k+2}}{3} + (t-1)2^k, & n(=2k+1) \text{ is odd} \end{cases}$$

(2) The graph C_n satisfies the $2t-$pebbling property.

This paper is organized as follows. In Section 2, we define the zig-zag chain graph of even cycles and give some preliminary results which are used in our main results. In Section 3, we state and prove our main results of pebbling on zig-zag chain graph of even cycles.

2. Preliminaries

We now define the zig-zag chain graph of even cycle and discuss some results which are useful for subsequent sections.

Definition 2.1. The *zig-zag chain graph of even cycles* denoted by ZZ_n, is a graph which consists of zig-zag sequence of n even cycles, C_{2k} with $k \geq 3$. We have the following vertex set and edge set of ZZ_n for n even as follows: $V(ZZ_n) = \{a_i, b_i : 1 \leq i \leq n(k-1)\} \cup \{x, y\}$ and

$E(ZZ_n) = \{a_i\, a_{i+1}, b_i b_{i+1} : 1 \le i \le n(k-1)-1\} \cup$
$\{xa_1, xb_1, ya_{n(k-1)}, yb_{n(k-1)}\} \cup$
$\left\{a_{i(k+1)-1} b_{i(k+1)-2}, a_{j(k+1)} b_{j(k+1)+1} : 1 \le i \le \frac{n}{2}, 1 \le j \le \frac{n-2}{2}\right\}$

For n odd, we have the following vertex set and edge set.

$V\,(ZZ_n) = \{a_i, b_i \,:\, 1 \le i \le n(k-1)\} \cup \{x, y\}$and

$E(ZZ_n) = \{a_i a_{i+1}, b_i b_{i+1} : 1 \le i \le n(k-1)-1\} \cup$
$\{xa_1, xb_1, ya_{n(k-1)}, yb_{n(k-1)}\} \cup$
$\left\{a_{i(k+1)-1} b_{i(k+1)-2}, a_{j(k+1)} b_{j(k+1)+1} : 1 \le i, j \le \frac{n-1}{2}\right\}$

The Figure 1.1 depicts the graph ZZ_3 for $k = 3$.

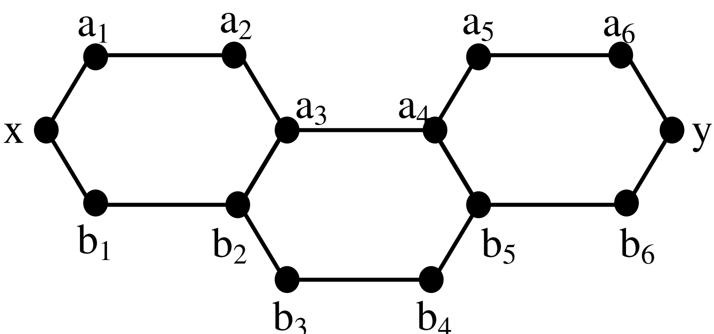

Figure 1.1.

The reader can easily view that ZZ_n has n copies of C_{2k}, and label each cycle as $C_1, C_2, \ldots,$ and C_n in order. Here, we present some results that will be used in the proof of main results.

Theorem 2.2. *[9] In a zig-zag chain graph of even cycles C_{2k}, denoted by ZZ_n with a specified vertex v, the following are true for $k \ge 3$.*

1. *If* $2^{n(k-1)+1}$ *pebbles are assigned to the vertices of the graph* ZZ_n, *one pebble can be moved to* v.
2. *Let* q *be the number of vertices with at least one pebble. If there are all together more than* $2(2^{n(k-1)+1}) - q$ *pebbles, then two pebbles can be moved to* v.

3. t −Pebbling number and $2t$ −Pebbling property

In this section, we find the t-pebbling number of the graph ZZ_n and show that ZZ_n satisfies the $2t$ −pebbling property by using induction.

Theorem 3.1. *Let* ZZ_n *be the zig-zag chain graph of* n *even cycles. We have*

$$f_t(ZZ_n) = t2^{n(k-1)+1} \; for \; n \geq 2 \; and \; k \geq 2.$$

Proof. We use induction on t. The result is true for $t = 1$ by Theorem 2.2. Suppose that the result is true for $t' < t$. Let D be a distribution of $t2^{n(k-1)+1}$ pebbles on the graph ZZ_n. Let $v \in C_t, 1 \leq t \leq n$, be any target vertex. We have to move t pebbles to v. Consider the following cases.

Case (1): Let $p_v = 0$.

We may move one pebble to v at a cost of at most $2^{n(k-1)+1}$ pebbles. Then remaining at least $(t - 1)2^{n(k-1)+1}$ pebbles are in ZZ_n. Thus by induction, we can move $(t - 1)$ additional pebbles to v.

Case (2): Let $p_v = x$, where $1 \leq x \leq t - 1$.

We have to move $(t - x)$ additional pebbles to v.

Now,
$$f_t(G) - x = t2^{n(k-1)+1} - x$$
$$= (t + x - x)2^{n(k-1)+1} - x$$
$$\geq (t - x)2^{n(k-1)+1}$$
$$= f_{(t-x)}(G)$$

By induction we can move $(t - x)$ additional pebbles to v.

Theorem 3.2. *The graph ZZ_n satisfies the $2t -$ pebbling property.*

Proof: We use induction on t. For $t = 1$, the result follows from Theorem 2.2. Assume for $t' < t$. Let D be the distribution of more than $2(f_t(G)) - q$ pebbles on the graph ZZ_n. We have to move $2t$ pebbles to any specified vertex. Consider the following cases.

Case (1): Let $p_v = 0$.

We can write
$$2\big(f_t(G)\big) - q + 1 = 2(f_{t-1}(G) + f(G)) - q + 1$$
$$= 2\big(f_{t-1}(G)\big) - q + 1 + 2\big(f(G)\big).$$

By using Theorem 3.1, we can put two pebbles to v by exactly using $2f(G)$ pebbles. We have remaining $2(f_{t-1}(G)) - q + 1$ pebbles, and so we can put $(t - 1)$ additional pebbles to v.

Case (2): Let $p_v = x$, where $1 \leq x \leq 2t - 1$.

We have to move $(2t - x)$ additional pebbles to v. Therefore we consider the following possibilities.

Subcase 2(*a*): x is even.

Take $x = 2x_1$, where x_1 any positive integer is.

We have

$$2(f_t(G)) - q + 1 + x = 2(f_{(t-x_1+x_1)}(G)) - q + 1 - 2x_1$$

$$= 2\left(f_{t-x_1}(G) + f_{x_1}(G)\right) - q + 1 - 2x_1$$

$$= 2\left(f_{t-x_1}(G)\right) - q + 1 + 2x_1(2^{n(k-1)+1} - 1)$$

$$\geq 2\left(f_{t-x_1}(G)\right) - q + 1$$

Thus we can move $(2t - x)$ additional pebbles to v.

Subcase 2(*b*): x is odd.

We can make given x as even by adding one pebble, using exactly $2^{n(k-1)+1}$ pebbles we can put one pebble to v. Now we have $x + 1 = 2x_1$ pebbles on v, where x_1 is any non-negative integer. Therefore we should move $2(t - x_1)$ additional pebbles to v.

Now, we have remaining at least $2(f_t(G)) - q + 1 - 2^{n(k-1)+1} - (x + 1)$

$$= 2(f_{t-1}(G) + f(G)) - q + 1 - f(G) - 2x_1$$

$$= 2(f_{t-1}(G)) - q + 1 + f(G) - 2x_1$$

$$= 2(f_{t-x_1}(G) - f(G) + f_{x_1}(G)) - q + 1 + f(G) - 2x_1$$

$$= 2(f_{t-x_1}(G)) - q + 1 - f(G) + 2f_{x_1}(G) - 2x_1$$

$\geq 2(f_{t-x_1}(G)) - q + 1$ pebbles.

Then we can put $2(t - x_1)$ additional pebbles to v.

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